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Series solution of the temperature distribution in the Falkner–Skan wedge flow by the homotopy analysis method

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ABSTRACT

A new analytical method, namely the homotopy analysis method (HAM), has been applied to investigate the temperature field associated with the Falkner–Skan boundary-layer problem, and a series solution is provided in this paper. The results of the present work show agreement with those of numerical solutions in a large range of Prandtl numbers ($0 < Pr \leq 100$), which demonstrates the validity of the present analysis.

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1. Introduction

The analysis of the temperature field, associated with a two-dimensional steady and incompressible laminar flow passing a wedge, governed by the famous Falkner–Skan equation [1], has extensive applications within the fields of aerodynamics and hydrodynamics. Researchers such as Hartree [2], Howarth [3], Asaithambi [4], Cebeci and Keller [5], and Sher and Yakhot [6], have numerically investigated the solutions of the Falkner–Skan equation owing to the difficulties in obtaining an exact solution to the problem considered in a closed form.

As far as the heat transfer analysis of the Falkner–Skan wedge flow is concerned, Lin and Lin [7] has introduced a similarity solution method for the forced convection heat transfer from isothermal or uniform-flux surfaces to fluids of any Prandtl number, and then solved the resulting similarity equations by the Runge–Kutta scheme. Hsu et al. [8] has studied the temperature and flow fields of the flow past a wedge by the series expansion method, Runge–Kutta integration and the shooting method. Kuo [9] has investigated the temperature field associated with the Falkner–Skan boundary-layer problem by converting it into a pair of initial value problems with the usage of the differential transformation method, and then calculating it numerically. In particular, Liao's analysis [10] has applied a new analytical method, the so-called homotopy analysis method, to give an analytical solution of the temperature distribution in viscous Blasius flow problems.

Different from previous work, the author investigates the temperature field associated with the Falkner–Skan wedge flow by the homotopy analysis method (HAM), and a series solution is given in this paper. This solution shows that the current results are in agreement with those provided by previous numerical methods, which proves the validity of the present work.

2. Mathematical description

For a main stream with velocity U varying as x^k, the transformations

$$\psi(x,y) = \left(\frac{2\nu xU}{k+1}\right)^{1/2} f(\eta), \quad \eta = y\left(\frac{U(k+1)}{2\nu x}\right)^{1/2}, \tag{1}$$

reduce the equations governing a two-dimensional steady and incompressible laminar flow passing a wedge to the famous nonlinear Falkner–Skan equation [1]

$$f'''(\eta) + f(\eta)f''(\eta) + \beta\left[1 - f'^2(\eta)\right] = 0, \tag{2}$$

with the boundary conditions

$$f(0) = 0, \quad f'(0) = 0, \quad f'(+\infty) = 1, \tag{3}$$

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where $\beta = 2k/(k+1)$ is a parameter of the stream-wise pressure gradient, $\psi(x, y)$is a stream function, x and y are coordinates along and normal to the boundary layer, and υ is the kinematic viscosity. The velocity components along and normal to the boundary layer are given by

$$u = Uf'(\eta), \quad v = [f(\eta) - (k-1)\eta f'(\eta)]\sqrt{\frac{\upsilon U}{2(k+1)x}}. \tag{4}$$

To consider the related heat transfer problem, a non-dimensional temperature is defined as

$$\theta = \frac{T_w - T}{T_w - T_\infty}, \tag{5}$$

where T denotes the dimensional temperature, and T_w and T_∞ are constant temperatures at the boundary and at infinity, respectively. Thus, in the absence of frictional heat, $\theta(\eta)$ satisfies the second-order differential equation

$$\theta''(\eta) + Pr\cdot f(\eta)\theta'(\eta) = 0, \tag{6}$$

with boundary conditions

$$\theta(0) = 0, \quad \theta(+\infty) = 1, \tag{7}$$

where Pr is the Prandtl number, i.e. the ratio of the momentum diffusivity of the fluid and its thermal diffusivity. Details can be found in Ref. [9].

3. Series solution to the temperature distribution in the Falkner–Skan wedge flow by HAM

Considering the boundary conditions in (3) and (7), it is reasonable to choose two sets of basis functions

$$\left\{\eta^m e^{-2n\eta}, \quad m, n \geq 0\right\}, \tag{8}$$

and

$$\left\{\eta^m e^{-n\eta}, \quad m, n \geq 0\right\}, \tag{9}$$

to express the solutions of Eqs. (2) and (6) as

$$f(\eta) = \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty} a_{m,n}\eta^m e^{-2n\eta}, \tag{10}$$

and

$$\theta(\eta) = \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty} b_{m,n}\eta^m e^{-n\eta}, \tag{11}$$

where $a_{m,n}$ and $b_{m,n}$ are coefficients.

3.1. Zero-order deformation equation of HAM

Considering the boundary conditions in (3) and (7) and the solution expressions in (10) and (11), the initial guess solutions of Eqs. (2) and (6) are chosen to be

$$f_0(\eta) = \eta - \frac{1 - e^{-2\eta}}{2}, \tag{12}$$

and

$$\theta_0(\eta) = 1 - e^{-\eta}, \tag{13}$$

and correspondingly the auxiliary linear operators $\mathcal{L}_f$ and $\mathcal{L}_\theta$ are chosen to be

$$\mathcal{L}_f[\phi(\eta; q)] = \frac{\partial^3\phi}{\partial\eta^3} - 4\frac{\partial\phi}{\partial\eta}, \tag{14}$$

and

$$\mathcal{L}_\theta[\phi(\eta; q)] = \frac{\partial^2\phi}{\partial\eta^2} + \frac{\partial\phi}{\partial\eta}, \tag{15}$$

which have the properties of

$$\mathcal{L}_f\left[C_1 e^{2\eta} + C_2 e^{-2\eta} + C_3\right] = 0, \tag{16}$$

and

$$\mathcal{L}_\theta\left[D_1 e^{-\eta} + D_2\right] = 0, \tag{17}$$

where C_1, C_2, C_3, D_1, and D_2 are integral constants to be determined by boundary conditions. Obviously, C_1 should be chosen to be zero because we seek a finite solution. Next, zero-order deformation equations of HAM are constructed as

$$\begin{aligned}(1-q)\mathcal{L}_f[\phi(\eta; q) - f_0(\eta)] = -qH_f(\eta)\Bigg\{&\frac{\partial^3\phi(\eta; q)}{\partial\eta^3} \\ &+\phi(\eta; q)\frac{\partial^2\phi(\eta; q)}{\partial\eta^2} \\ &+\beta\left[1 - \left(\frac{\partial\phi(\eta; q)}{\partial\eta}\right)^2\right]\Bigg\},\end{aligned} \tag{18}$$

and

$$\begin{aligned}(1-q)\mathcal{L}_\theta[\Theta(\eta; q) - \theta_0(\eta)] = q\hbar H_\theta(\eta)\Bigg[&\frac{\partial^2\Theta(\eta; q)}{\partial\eta^2} \\ &+ Pr\cdot\phi(\eta; q)\frac{\partial\Theta(\eta; q)}{\partial\eta}\Bigg],\end{aligned} \tag{19}$$

with the boundary conditions

$$\begin{aligned}&\phi(0; q) = 0, \ \left.\frac{\partial\phi(\eta; q)}{\partial\eta}\right|_{\eta=0} = 0, \ \left.\frac{\partial\phi(\eta; q)}{\partial\eta}\right|_{\eta=+\infty} = 1, \ \Theta(0; q) \\ &\quad = 0, \ \Theta(+\infty; q) = 1,\end{aligned} \tag{20}$$

where $q \in [0, 1]$ is an embedding parameter, $\hbar$ is a non-zero auxiliary parameter, and $H_f(\eta)$ and $H_\theta(\eta)$ are non-zero auxiliary functions.

Clearly, when $q = 0$, the solutions to Eqs. (18) and (19) are given by

$$\phi(\eta; 0) = f_0(\eta), \quad \Theta(\eta; 0) = \theta_0(\eta), \tag{21}$$

and when $q = 1$, because of $\hbar \neq 0$, $H_f(\eta) \neq 0$ and $H_\theta(\eta) \neq 0$ the solutions are equivalent to those of Eqs. (2) and (6) provided that the conditions

$$\phi(\eta; 1) = f(\eta), \tag{22}$$

and

$$\Theta(\eta; 1) = \theta(\eta), \tag{23}$$

are satisfied. Therefore, as the embedding parameter q increases from 0 to 1, $\phi(\eta; q)$ varies continuously from the initial guess solutions $f_0(\eta)$ to the exact solutions $f(\eta)$ of Eq. (2), and so does $\Theta(\eta; q)$ from the initial guess solution $\theta_0(\eta)$ to the exact solutions $\theta(\eta)$ of Eq. (6). The process is the so-called deformation in topology. With the aid of terms used in topology, Eqs. (18) and (19) are correspondingly called zero-order deformation equations of HAM, and detailed descriptions can be found in Ref. [11]. Therefore, $\phi(\eta; q)$ and $\Theta(\eta; q)$ can be expanded in a Taylor series with respect to q to provide

$$\phi(\eta;q) = f_0(\eta) + \sum_{k=1}^{+\infty} f_k(\eta) q^k, \tag{24}$$

and

$$\Theta(\eta;q) = \theta_0(\eta) + \sum_{k=1}^{+\infty} \theta_k(\eta) q^k, \tag{25}$$

where

$$f_k(\eta) = \frac{1}{k!} \frac{\partial^k \phi(\eta;q)}{\partial q^k}\bigg|_{q=0}, \tag{26}$$

and

$$\theta_k(\eta) = \frac{1}{k!} \frac{\partial^k \Theta(\eta;q)}{\partial q^k}\bigg|_{q=0}. \tag{27}$$

Obviously, the convergence region of the series (24) and (25) depends upon the auxiliary linear operators $\mathcal{L}_f$ and $\mathcal{L}_\theta$, the auxiliary parameter $\hbar$, and the two non-zero auxiliary functions $H_f(\eta)$ and $H_\theta(\eta)$. If all of them are properly chosen so that the convergences of the above two series at $q = 1$ are guaranteed, the series solutions of Eqs. (2) and (6) can be expressed as follows

$$\phi(\eta;q) = f_0(\eta) + \sum_{k=1}^{+\infty} f_k(\eta), \tag{28}$$

$$\Theta(\eta;q) = \theta_0(\eta) + \sum_{k=1}^{+\infty} \theta_k(\eta). \tag{29}$$

3.2. *k*th-Order deformation equation of HAM

For convenience, define the vectors

$$\overrightarrow{f_k} = \{f_0(\eta), f_1(\eta), \ldots, f_k(\eta)\}, \tag{30}$$

$$\overrightarrow{\theta_k} = \{\theta_0(\eta), \theta_1(\eta), \ldots, \theta_k(\eta)\}, \tag{31}$$

where $k \in \mathbf{N}$. Differentiating the zero-order deformation Eqs. (18) and (19) k times with respect to q, setting $q = 0$, and then dividing by $k!$, the kth-order deformation equation is obtained by

$$\mathcal{L}_f[f_k(\eta) - \chi_k f_{k-1}(\eta)] = -H_f(\eta) R_k^f(\overrightarrow{f}_{k-1}, \eta), \tag{32}$$

$$\mathcal{L}_\theta[\theta_k(\eta) - \chi_k \theta_{k-1}(\eta)] = \hbar H_\theta(\eta) R_k^\theta(\overrightarrow{f}_{k-1}, \overrightarrow{\theta}_{k-1}, \eta), \tag{33}$$

with the boundary conditions

$$f_k(0) = 0, \quad f_k'(0) = 0, \quad f_k'(+\infty) = 0, \quad \theta_k(0) = 0, \quad \theta_k(+\infty) = 0, \tag{34}$$

where

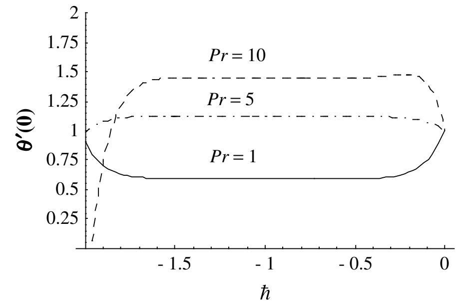

Fig. 1. Graph showing $\theta'(0)$ as a function of $\hbar$ for the 10th-order analytical approximation solution with $\beta = 2$ for $Pr = 1$, 5, and 10.

$$\chi_k = \begin{cases} 0, & k \le 1 \\ 1, & k > 1 \end{cases}, \tag{35}$$

$$R_k^f(\overrightarrow{f}_{k-1}, \eta) = f_{k-1}'''(\eta) + \sum_{j=0}^{k-1} \left[f_j(\eta) f_{k-1-j}''(\eta) - \beta f_j'(\eta) f_{k-1-j}'(\eta) \right] + \beta(1 - \chi_k), \tag{36}$$

$$R_k^\theta(\overrightarrow{f}_{k-1}, \overrightarrow{\theta}_{k-1}, \eta) = \theta_{k-1}''(\eta) + Pr \cdot \sum_{j=0}^{k-1} f_j(\eta) \theta_{k-1-j}'(\eta). \tag{37}$$

Let $f_k^*(\eta)$ and $\theta_k^*(\eta)$ denote special solutions of equations

$$\mathcal{L}_f\left[f_k^*(\eta)\right] = -H_f(\eta) R_k^f(\overrightarrow{f}_{k-1}, \eta), \tag{38}$$

$$\mathcal{L}_\theta\left[\theta_k^*(\eta)\right] = \hbar H_\theta(\eta) R_k^\theta(\overrightarrow{f}_{k-1}, \overrightarrow{\theta}_{k-1}, \eta), \tag{39}$$

and then according to the properties of linear operators in Eqs. (16) and (17), we seek the solutions to Eqs. (32) and (33) in the following form

$$f_k(\eta) = \chi_k f_{k-1}(\eta) + f_k^*(\eta) + C_1^k e^{2\eta} + C_2^k e^{-2\eta} + C_3^k, \tag{40}$$

$$\theta_k(\eta) = \chi_k \theta_{k-1}(\eta) + \theta_k^*(\eta) + D_1^k e^{-\eta} + D_2^k, \tag{41}$$

where C_1^k, C_2^k and C_3^k are coefficients to be determined by the boundary conditions in (34), i.e.

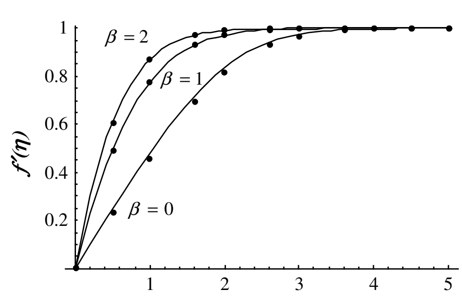

Fig. 2. Comparison of the 15th order HAM solution of $f'(\eta) \sim \eta$ with numerical results by White [12] for $\beta = 0$, 1, and 2. The dots represent the numerical results of White [12].

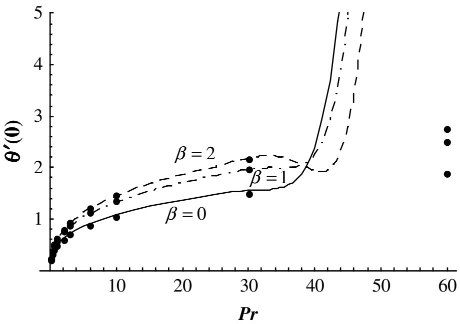

Fig. 3. Comparison of the 10th-order HAM solution of $\theta'(0)$ with numerical results for $\hbar = -1/2$ for $\beta = 0$, 1, and 2. The dots represent the numerical results of White [12].

$$C_1^k = 0,\ C_2^k = f_k^{*\prime}(0)/2,\ C_3^k = -f_k^*(0) - f_k^{*\prime}(0)/2,\ D_1^k = -\theta_k^*(0),\quad D_2^k = 0. \tag{42}$$

Furthermore, according to the rules of the solution expression and coefficient ergodicity by the homotopy analysis method [11], and due to the solution expressions of Eqs. (10) and (11), the auxiliary functions $H_f(\eta)$ and $H_\theta(\eta)$ are uniquely determined by

$$H_f(\eta) = 1, \quad H_\theta(\eta) = e^{-\eta}. \tag{43}$$

Thus, the nonlinear Eqs. (2) and (6) are converted into a series of linear boundary value problems as in Eqs. (32) and (33), which can be easily solved by symbolic computation software such as Mathematica and Maple.

4. Results and analysis

As Liao [11] has proved that, as long as a series solution given by the homotopy analysis method converges, it must be one of the solutions for the problem considered. Therefore, we can provide an approximate analytical solution to the heat transfer analysis of the Falkner–Skan wedge flow by a finite number of terms of infinite series within the admission of errors. It should be emphasized once again that one has freedom to choose proper basis functions to approximate a nonlinear problem, and that better basis functions help to accelerate the convergence of the series solution, while inappropriate choices of basis functions will result in poor properties of convergence or even divergence of the series solution. This is the reason why different basis functions are applied in Eqs. (10) and (11). The chosen basis functions also determine the choices of initial guess

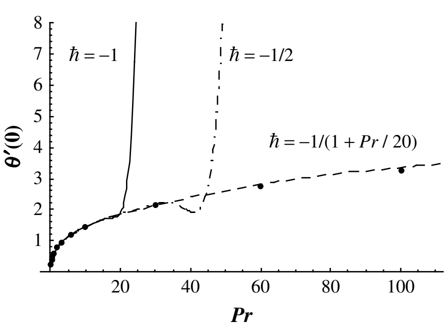

Fig. 4. The 10th-order HAM solution of $\theta'(0)$ with different $\hbar$ for $\beta = 2$ (solid line: $\hbar = -1$; dotted line: $\hbar = -1/2$; dashed line: $\hbar = -1/(1 + Pr/20)$). The dots represent the numerical results of White [12].

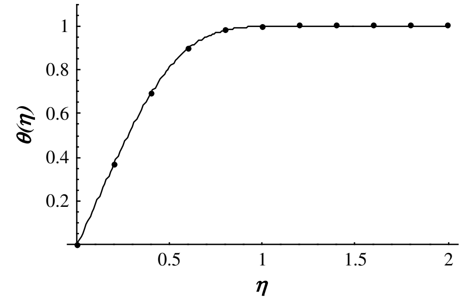

Fig. 5. The analytical solution of $\theta(\eta)$for the heat analysis of the Falkner–Skan wedge flow with $\beta = 2$, $Pr = 20$ and $\hbar = -1/2$. The solid line represents the 30th order HAM approximate solution, and the dots the 20th order HAM approximate solution.

solutions and auxiliary linear operators, and one can easily solve the linear kth-order deformation Eqs. (32) and (33) by expressing the solutions of the problem considered in terms of the combination of basis functions in Eqs. (8) and (9). This is the so-called solution expression rule in the homotopy analysis method [11].

Consider that the solution series (29) contains the auxiliary parameter $\hbar$, which can be determined by plotting the so-called $\theta'(0) \sim \hbar$ curve as suggested by Liao [11]. For instance, the proper $\hbar$ can be identified according to the 10th-order analytical approximation solution curve of $\theta'(0) \sim \hbar$, where some fixed Pr is evaluated to be 1, 5 and 10, respectively, with the fixed parameter of $\beta = 2$, as shown in Fig. 1. According to the convergent region of the $\theta'(0) \sim \hbar$ curve, one can choose the proper $\hbar$.

A comparison of the HAM solution $f'(\eta) \sim \eta$ with the numerical results by White [12] is shown in Fig. 2. Moreover, we also compare the 10th-order HAM solution of $\theta'(0)$ with numerical results of White [12] for $\hbar = -1/2$, where Pr varies from 0 to 100, and β is evaluated to be 0, 1 and 2, respectively, as shown in Fig. 3. It is shown that the present work agrees very well with that of White [12] for $0 < Pr \leq 30$.

Interestingly, the choice of $\hbar$ provides an easy way to enlarge the convergent region of the series solution to the heat transfer analysis of the Falkner–Skan wedge flow. Fig. 4 shows how the convergent region of $\theta'(0)$ varies with different $\hbar$ values when $\beta = 2$. Especially, when $\hbar = -1/(1 + Pr/20)$, the convergent region is extended to a very large range of Prandtl numbers: $0 < Pr \leq 100$.

We also give the 20th and 30th order approximate analytical solutions of $\theta(\eta)$and $\theta'(\eta)$ for the heat transfer analysis of the Falkner–Skan wedge flow with $\beta = 2$, $Pr = 20$ and $\hbar = -1/2$, as shown in

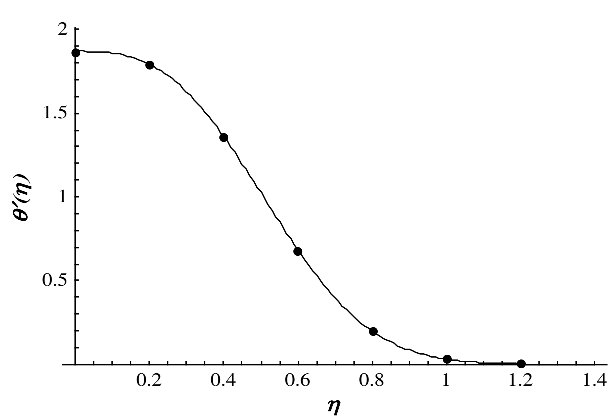

Fig. 6. Graph showing $\theta'(\eta)$ as a function of η for the heat analysis of the Falkner–Skan wedge flow with $\beta = 2$, $Pr = 20$ and $\hbar = -1/2$. The solid line represents the 30th order HAM approximate solution, and the dots the 20th order HAM approximate solution.

Figs. 5 and 6. This further shows the convergence of the series solution to the heat transfer analysis of the Falkner–Skan wedge flow according to the homotopy analysis method.

5. Conclusions

A new analytical method, namely the homotopy analysis method (HAM), has been applied to investigate the temperature field associated with the Falkner–Skan boundary-layer problem and a series solution is given in this paper. The results of the present work agree well with those of the numerical method by White [12]. Interestingly, the choice of $\hbar$ provides an easy way to extend the convergent region of the series solution to heat transfer analysis of the Falkner–Skan wedge flow. The results of the present work are effective for a very large range of Prandtl numbers ($0 < Pr \leq 100$), which shows the validity of the present work.

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